The Derivative

Definition of the Derivative

The derivative of the function *f*(*x*) at the point x_0 is given and denoted by

$$\frac{df}{dx}(x_0) = f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Some Basic Derivatives

In the table below, *u*,*v*, and *w* are functions of the variable *x*. *a*, *b*, *c*, and *n* are constants (with some restrictions whenever they apply). $\ln(x)$ designate the natural logarithmic function and *e* the natural base for $\ln(x)$. Recall that $e = 2.718\cdots$.

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(c\,u) = c\,\frac{du}{dx}$$

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(u \circ v) = \frac{dv}{dx}\,\left(\frac{du}{dx} \circ v\right)$$

Chain Rule

The last formula

$$\frac{d}{dx}(u \circ v) = \frac{dv}{dx}\,\left(\frac{du}{dx} \circ v\right)$$

is known as the Chain Rule formula. It may be rewritten as

$$\frac{du}{dx} = \frac{du}{dv}\,\frac{dv}{dx}$$

Another similar formula is given by

$$\frac{du}{dx} = \frac{\frac{du}{dv}}{\frac{dx}{dv}}$$

Derivative of the Inverse Function

The inverse of the function $y(x)$ is the function $x(y)$, we have

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Derivative of Trigonometric Functions and their Inverses

$$\frac{d}{dx}(\sin(u)) = \cos(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\cos(u)) = -\sin(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\tan(u)) = \sec^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\cot(u)) = -\csc^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\sec(u)) = \sec(u)\tan(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\csc(u)) = -\csc(u)\cot(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cos^{-1}(u)) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\tan^{-1}(u)) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}(\cot^{-1}(u)) = -\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}(\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2 \quad 1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\csc^{-1}(u)) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Recall the definitions of the trigonometric functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}$$

Derivative of the Exponential and Logarithmic functions

$$\frac{d}{dx}(\ln(u)) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a(u)) = \frac{1}{\ln(a)u}\frac{du}{dx}$$

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

$$\frac{d}{dx}(a^u) = \ln(a)a^u\frac{du}{dx}$$

$$\frac{d}{dx}(u^v) = \frac{d}{dx}\left(e^{v\ln(u)}\right) = e^{v\ln(u)}\frac{du}{dx}\left(v\ln(u)\right) = vu^{v-1}\frac{du}{dx} + u^v\ln(u)\frac{dv}{dx}$$

Recall the definition of the logarithm function with base *a* > 0 (with $a \neq 1$):

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Derivative of the Hyperbolic functions and their Inverses

$$\frac{d}{dx}(\sinh(u)) = \cosh(u)\frac{du}{dx}$$

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$$\frac{d}{dx}(\tanh(u)) = \operatorname{sech}^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\coth(u)) = -\operatorname{csch}^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech}(u)) = -\operatorname{sech}(u)\tanh(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch}(u)) = -\operatorname{csch}(u)\coth(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\sinh^{-1}(u)) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1}(u)) = \frac{\pm 1}{\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\tanh^{-1}(u)) = \frac{\pm 1}{1-u^2}\frac{du}{dx} \qquad (-1 < u < 1)$$

$$\frac{d}{dx}(\coth^{-1}(u)) = \frac{1}{1-u^2}\frac{du}{dx} \qquad (u > 1 \text{ or } u < -1)$$

$$\frac{d}{dx}(\operatorname{sech}^{-1}(u)) = -\frac{\pm 1}{|u|\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch}^{-1}(u)) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

Recall the definitions of the trigonometric functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}, \quad \text{sech}(x) = \frac{1}{\cosh(x)}, \quad \text{csch}(x) = \frac{1}{\sinh(x)}$$

Higher Order Derivatives

Let $y = f(x)$. We have:

$$\textbf{Second Derivative is:} \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$$

$$\textbf{Third Derivative is:} \quad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$$

$$n^{\text{th}} \textbf{ Derivative is:} \quad \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$$

In some books, the following notation for higher derivatives is also used:

$$D^n(y) = \frac{d^ny}{dx^n}$$

Higher Derivative Formula for the Product: Leibniz Formula

$$\frac{d^n}{dx^n}(u\,v) = D^n(u\,v) = uD^n(v) + \binom{n}{1} D(u)D^{n-1}(v) + \cdots + \binom{n}{k} D^k(u)D^{n-k}(v) + \cdots + vD^n(u)$$

where $\binom{n}{k}$ are the binomial coefficients. For example, we have

$$\frac{d^2}{dx^2}\left(u\ v\right) = u\frac{d^2 v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2 u}{dx^2}$$

$$\frac{d^3}{dx^3}\left(u\ v\right) = u\frac{d^3 v}{dx^3} + 3\frac{du}{dx}\frac{d^2 v}{dx^2} + 3\frac{dv}{dx}\frac{d^2 u}{dx^2} + v\frac{d^3 u}{dx^3}$$